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# Asymptotic behavior of bivariate Gaussian powered extremes

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A B S T R A C T

In this paper, joint asymptotics of powered maxima for a triangular array of bivariate Gaussian random vectors are considered. Under the Hüsler–Reiss condition, limiting distributions of powered maxima are derived. Furthermore, the second-order expansions of the joint distributions of powered maxima are established under the refined Hüsler–Reiss condition.



## 1. Introduction

For independent and identically distributed bivariate Gaussian random vectors with constant coefficient in each vector, Sibuya [16] showed that componentwise maxima are asymptotically independent, and Embrechts et al. [1] proved the asymptotical independence in the upper tail. To overcome those shortcomings in its applications, Hüsler and Reiss [11] considered the asymptotic behaviors of extremes of Gaussian triangular arrays with varying coefficients. Precisely, let $\{(X_{ni}, Y_{ni}), 1 \leq i \leq n, n \geq 1\}$ be a triangular array of independent bivariate Gaussian random vectors with $\mathrm{E}\, X_{ni} = \mathrm{E}\, Y_{ni} = 0$, $\mathrm{Var}\, X_{ni} = \mathrm{Var}\, Y_{ni} = 1$ for $1 \leq i \leq n$, $n \geq 1$. and $\mathrm{Cov}(X_{ni}, Y_{ni}) = \rho_n$. Let $F_{\rho_n}(x, y)$ denote the joint distribution of vector $(X_{ni}, Y_{ni})$ for $i \leq n$. The partial maxima $M_n$ is defined by

$$M_n = (M_{n1}, M_{n2}) = (\max_{1\leq i\leq n} X_{ni}, \max_{1\leq i\leq n} Y_{ni}).$$

Hüsler and Reiss [11] and Kabluchko et al. [12] showed that

$$\lim_{n\to\infty} \mathbb{P}\left(M_{n1} \leq b_n + \frac{x}{b_n}, M_{n2} \leq b_n + \frac{y}{b_n}\right) = H_\lambda(x, y) \tag{1.1}$$

holds if and only if the following Hüsler–Reiss condition

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$$\lim_{n\to\infty} b_n^2(1-\rho_n) = 2\lambda^2 \in [0,\infty] \tag{1.2}$$

holds, where the normalizing constant $b_n$ is the solution of the equation

$$1-\Phi(b_n) = n^{-1} \tag{1.3}$$

and the max-stable Hüsler–Reiss distribution is given by

$$H_\lambda(x,y) = \exp\left(-\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y} - \Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\right), \quad x,y\in\mathbb{R}, \tag{1.4}$$

where $\Phi(\cdot)$ denotes the distribution function of a standard Gaussian random variable. Note that $H_0(x,y) = \Lambda(\min(x,y))$ and $H_\infty(x,y) = \Lambda(x)\Lambda(y)$ with $\Lambda(x) = \exp(-\exp(-x))$, $x\in\mathbb{R}$.

Recently, contributions to Hüsler–Reiss distribution and its extensions are achieved considerably. For instance, Hashorva [4,5] showed that the limit distributions of maxima also hold for triangular arrays of general bivariate elliptical distributions if the distribution of random radius is in the Gumbel or Weibull max-domain of attraction, and Hashorva and Ling [9] extended the results to bivariate skew elliptical triangular arrays. For more work on asymptotics of bivariate triangular arrays, see [6–8].

Higher-order expansions of distributions of extremes on Hüsler–Reiss bivariate Gaussian triangular arrays were considered firstly by Hashorva et al. [10] provided that $\rho_n$ satisfies the following refined Hüsler–Reiss condition

$$\lim_{n\to\infty} b_n^2(\lambda_n-\lambda) = \alpha\in\mathbb{R}, \tag{1.5}$$

where $\lambda_n = (\frac{1}{2}b_n^2(1-\rho_n))^{1/2}$ and $\lambda\in(0,\infty)$, with $b_n$ given by (1.3). Uniform convergence rate was considered by Liao and Peng [14]. For copula version of the limit in Hüsler–Reiss model, Frick and Reiss [2] considered the penultimate and ultimate convergence rates for distribution of $(n(\max_{1\le i\le n}\Phi(X_{ni})-1), n((\max_{1\le i\le n}\Phi(Y_{ni})-1)))$, and Liao et al. [15] extended the results to the settings of $n$ independent and non-identically distributed observations, where the $i$th observation follows from normal copula with correlation coefficient being either a parametric or a nonparametric function of $i/n$.

The objective of this paper is to study the asymptotics of powered-extremes of Hüsler–Reiss bivariate Gaussian triangular arrays. Interesting results in [3] showed that the convergence rates of the distributions of powered-extremes of independent and identically distributed univariate Gaussian sequence depend on the power index and normalizing constants. Precisely, let $|M_n|^t$ denote the powered maximum with any power index $t>0$, then

$$\lim_{n\to\infty} b_n^2\left[\mathbb{P}\left(|M_n|^t \le c_n x + d_n\right) - \Lambda(x)\right] = \Lambda(x)\mu(x) \tag{1.6}$$

with normalizing constants $c_n$ and $d_n$ given by

$$c_n = tb_n^{t-2}, \quad d_n = b_n^t, \quad t>0. \tag{1.7}$$

Furthermore, for $t=2$ with normalizing constants $c_n^*$ and $d_n^*$ given by

$$c_n^* = 2-2b_n^{-2}, \quad d_n^* = b_n^2 - 2b_n^{-2}, \tag{1.8}$$

we have

$$\lim_{n\to\infty} b_n^4\left[\mathbb{P}\left(|M_n|^2 \le c_n^* x + d_n^*\right) - \Lambda(x)\right] = \Lambda(x)\nu(x), \tag{1.9}$$

where $b_n$ is defined by (1.3), and $\mu(x)$ and $\nu(x)$ are respectively given by

$$\mu(x) = \left(1 + x + \frac{2-t}{2}x^2\right)e^{-x}, \quad \nu(x) = -\left(\frac{7}{2} + 3x + x^2\right)e^{-x}. \tag{1.10}$$

Motivated by findings of Hüsler–Reiss [11], Hall [3] and Hashorva et al. [10], we will consider the distributional asymptotics of powered-extremes of Hüsler–Reiss bivariate Gaussian triangular arrays, and hope that the convergence rates can be improved as $t = 2$, similar to (1.9) in univariate case. Unfortunately, our results provide negative answers except two extreme cases.

The rest of the paper is organized as follows. In Section 2 we provide the main results and all proofs are deferred to Section 4. Some auxiliary results are given in Section 3.

## 2. Main results

In this section, the limiting distributions and the second-order expansions on distributions of normalized bivariate powered-extremes are provided if $\rho_n$ satisfies (1.2) and (1.5), respectively. The first main result, stated as follows, is the limit distribution of bivariate normalized powered-extremes.

**Theorem 2.1.** *If the Hüsler–Reiss condition (1.2) holds with $\lambda \in (0, \infty)$, then for all $x, y \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(|M_{n1}|^t \le c_n x + d_n, |M_{n2}|^t \le c_n y + d_n\right) = H_\lambda(x, y), \tag{2.1}$$

*where the normalizing constants $c_n$ and $d_n$ are given by (1.7).*

**Remark 2.1.** For $t = 2$, with arguments similar to the proof of Theorem 2.1 one can show that (2.1) also holds with $c_n$ and $d_n$ being replaced by $c_n^*$ and $d_n^*$ given by (1.8).

Next we investigate the convergence rate of

$$\Delta(F^n_{\rho_n,t}, H_\lambda; c_n, d_n; x, y) = \mathbb{P}\left(|M_{n1}|^t \le c_n x + d_n, |M_{n2}|^t \le c_n y + d_n\right) - H_\lambda(x, y) \to 0 \tag{2.2}$$

as $n \to \infty$ under the refined second-order Hüsler–Reiss condition (1.5).

**Theorem 2.2.** *If the second-order Hüsler–Reiss condition (1.5) holds with $\lambda_n = (\frac{1}{2}b_n^2(1-\rho_n))^{1/2}$ and $\lambda \in (0, \infty)$, then for all $x, y \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} (\log n)\Delta(F^n_{\rho_n,t}, H_\lambda; c_n, d_n; x, y) = \frac{1}{2}\tau(\alpha, \lambda, x, y, t)H_\lambda(x, y) \tag{2.3}$$

*with*

$$\tau(\alpha, \lambda, x, y, t) = \mu(x)\Phi\left(\lambda + \frac{y-x}{2\lambda}\right) + \mu(y)\Phi\left(\lambda + \frac{x-y}{2\lambda}\right) + \left(2\alpha - (x+y+2)\lambda - \lambda^3\right)e^{-x}\varphi\left(\lambda + \frac{y-x}{2\lambda}\right),$$

*where $\varphi(\cdot)$ denotes the density function of a standard Gaussian random variable and $\mu(x)$ is given by (1.10).*

**Remark 2.2.** For $t = 2$, let $c_n$ and $d_n$ be replaced by $c_n^*$ and $d_n^*$ respectively in (1.8), one can show that

$$\lim_{n\to\infty} (\log n)\Delta(F^n_{\rho_n,2}, H_\lambda; c_n^*, d_n^*; x, y) = \frac{1}{2}\chi(\alpha, \lambda, x, y)H_\lambda(x, y)$$

with

$$\chi(\alpha,\lambda,x,y)=\Big(2\alpha-(x+y+2)\lambda-\lambda^3\Big)e^{-x}\varphi\left(\lambda+\frac{y-x}{2\lambda}\right).$$

The result shows the fact that the convergence rates can not be improved as $t=2$ with normalizing constants $c_n^*$ and $d_n^*$, contrary to the result of univariate Gaussian case provided by Hall [3].

In order to obtain the convergence rates of (2.2) for two extreme cases $\lambda=0$ and $\lambda=\infty$, we need some additional conditions. The following results show that rates of convergence are considerably different with different choice of normalizing constants. For the case of $\lambda=\infty$, with normalizing constants $c_n$ and $d_n$ given by (1.7) we have the following results.

**Theorem 2.3.** *With all $x,y\in\mathbb{R}$, power index $t>0$ and the normalizing constants $c_n$ and $d_n$ given by (1.7). For $\rho_n\in[-1,1)$,*

(i) *if $\rho_n\in[-1,0]$, we have*

$$\lim_{n\to\infty}(\log n)\Delta(F^n_{\rho_n,t},H_\infty;c_n,d_n;x,y)=\frac{1}{2}\Big(\mu(x)+\mu(y)\Big)H_\infty(x,y), \tag{2.4}$$

(ii) *if $\rho_n\in(0,1)$ and $\lim_{n\to\infty}\frac{\log b_n}{b_n^2(1-\rho_n)}=0$, then (2.4) also holds.*

For the case of $\lambda=0$, we have the following results.

**Theorem 2.4.** *With $x,y\in\mathbb{R}$, power index $t>0$ and the normalizing constants $c_n$ and $d_n$ given by (1.7). For $\rho_n\in(0,1]$,*

(i) *if $\rho_n\equiv 1$, we have*

$$\lim_{n\to\infty}(\log n)\Delta(F^n_{\rho_n,t},H_0;c_n,d_n;x,y)=\frac{1}{2}\mu\Big(\min(x,y)\Big)H_0(x,y), \tag{2.5}$$

(ii) *if $\rho_n\in(0,1)$ and $\lim_{n\to\infty}b_n^6(1-\rho_n)=c_1\in[0,\infty)$, then (2.5) also holds.*

Theorems 2.3–2.4 show that convergence rates of (2.2) are the same order of $1/\log n$ if we choose the normalizing constants $c_n$ and $d_n$ given by (1.7). With another pair of normalizing constants $c_n^*$ and $d_n^*$ given by (1.8), the following results show that convergence rates of (2.2) can be improved. Theorem 2.5 stated below is for the case of $\lambda=\infty$.

**Theorem 2.5.** *With all $x,y\in\mathbb{R}$, power index $t=2$ and the normalizing constants $c_n^*$ and $d_n^*$ given by (1.8). For $\rho_n\in[-1,1)$,*

(i) *if $\rho_n\in[-1,0]$, we have*

$$\lim_{n\to\infty}(\log n)^2\Delta(F^n_{\rho_n,2},H_\infty;c_n^*,d_n^*;x,y)=\frac{1}{4}\Big(\nu(x)+\nu(y)\Big)H_\infty(x,y), \tag{2.6}$$

(ii) *if $\rho_n\in(0,1)$ and $\lim_{n\to\infty}\frac{\log b_n}{b_n^2(1-\rho_n)}=0$, then (2.6) also holds.*

For the case of $\lambda=0$, we have the following results.

**Theorem 2.6.** *With all $x, y \in \mathbb{R}$, power index $t = 2$ and normalizing constants $c_n^*$ and $d_n^*$ given by (1.8). For $\rho_n \in (0, 1]$,*

(i) *if $\rho_n \equiv 1$, we have*

$$\lim_{n\to\infty} (\log n)^2 \Delta(F^n_{\rho_n,2}, H_0; c_n^*, d_n^*; x, y) = \frac{1}{4}\nu\Big(\min(x, y)\Big)H_0(x, y), \tag{2.7}$$

(ii) *if $\rho_n \in (0, 1)$ and $\lim_{n\to\infty} b_n^{14}(1 - \rho_n) = c_2 \in [0, \infty)$, then (2.7) also holds.*

## 3. Auxiliary lemmas

For notational simplicity, let

$$\omega_{n,t}(x) = (c_n x + d_n)^{1/t} \quad \text{for} \quad t > 0, \quad \text{and} \quad \omega^*_{n,2}(x) = (c_n^* x + d_n^*)^{1/2} \quad \text{as} \quad t = 2, \tag{3.1}$$

where the normalizing constants $c_n$ and $d_n$, and $c_n^*$ and $d_n^*$ are those given by (1.7) and (1.8), respectively. Define

$$\bar{\Phi}(z) = 1 - \Phi(z), \qquad \bar{\Phi}_{n.t}(z) = n\bar{\Phi}(\omega_{n,t}(z))$$

and

$$I_k := \int_y^\infty \varphi\left(\lambda + \frac{x - z}{2\lambda}\right) e^{-z} z^k dz, \quad k = 0, 1, 2. \tag{3.2}$$

**Lemma 3.1.** *Under the conditions of Theorem 2.1, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(M_{n1} \le \omega_{n,t}(x), M_{n2} \le \omega_{n,t}(y)\right) = H_\lambda(x, y). \tag{3.3}$$

**Proof.** With the choice of $c_n$ and $d_n$ in (1.7), it follows from (3.1) that

$$\omega_{n,t}(z) = (c_n z + d_n)^{1/t} = b_n\left(1 + z b_n^{-2} + \frac{1-t}{2} z^2 b_n^{-4} + O(b_n^{-6})\right)$$

for fixed $z$, hence for fixed $x$ and $z$,

$$\begin{aligned}
&\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \\
&= b_n\sqrt{\frac{1-\rho_n}{1+\rho_n}} + \frac{x - z}{b_n\sqrt{1-\rho_n^2}} + \frac{z}{b_n}\sqrt{\frac{1-\rho_n}{1+\rho_n}} + \frac{(1-t)(x^2 - z^2)}{2b_n^3\sqrt{1-\rho_n^2}} + \frac{(1-t)z^2}{2b_n^3}\sqrt{\frac{1-\rho_n}{1+\rho_n}} + \sqrt{\frac{1-\rho_n}{1+\rho_n}}O(b_n^{-5}) \\
&= \left(\lambda_n + \frac{x-z}{2\lambda_n}\left(1 + \frac{(1-t)(x+z)}{2b_n^2}\right) + \frac{\lambda_n z}{b_n^2} + \frac{(1-t)\lambda_n z^2}{2b_n^4} + \lambda_n O(b_n^{-6})\right)\left(1 - \frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}}
\end{aligned} \tag{3.4}$$

which implies that

$$\lim_{n\to\infty} \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} = \lambda + \frac{x - z}{2\lambda} \tag{3.5}$$

holds since $\lambda_n \to \lambda$ as $n \to \infty$.

With $a_n = 1/b_n$ it follows from (3.5) that

$$
\begin{aligned}
& n\,\mathbb{P}\left(X > \omega_{n,t}(x), Y > \omega_{n,t}(y)\right) \\
&= n \int_{\omega_{n,t}(y)}^{\infty} \bar{\Phi}\left(\frac{\omega_{n,t}(x) - \rho_n z}{\sqrt{1-\rho_n^2}}\right) d\Phi(z) \\
&= \frac{b_n}{\varphi(b_n)}(1 - b_n^{-2} + O(b_n^{-4}))^{-1} \int_{y}^{\infty} \bar{\Phi}\left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \varphi(b_n(1+tza_n^2)^{1/t}) d(b_n(1+tza_n^2)^{1/t}) \\
&= (1 + b_n^{-2} + O(b_n^{-4})) \int_{y}^{\infty} \bar{\Phi}\left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \exp\left(\frac{b_n^2}{2}\left(1 - (1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1} dz \\
&\to \int_{y}^{\infty} \bar{\Phi}\left(\lambda + \frac{x-z}{2\lambda}\right) e^{-z} dz \\
&= e^{-y} + e^{-x} - \Phi\left(\lambda + \frac{x-y}{2\lambda}\right) e^{-y} - \Phi\left(\lambda + \frac{y-x}{2\lambda}\right) e^{-x} \qquad (3.6)
\end{aligned}
$$

as $n \to \infty$. Meanwhile, one can check that

$$\lim_{n\to\infty} \bar{\Phi}_{n,t}(x) = e^{-x}. \qquad (3.7)$$

It follows from (3.6) and (3.7) that

$$
\begin{aligned}
& \mathbb{P}\left(M_{n1} \le \omega_{n,t}(x), M_{n2} \le \omega_{n,t}(y)\right) \\
&= \exp\Big[ - \bar{\Phi}_{n,t}(x) - \bar{\Phi}_{n,t}(y) + n\,\mathbb{P}\left(X > \omega_{n,t}(x), Y > \omega_{n,t}(y)\right) + o(1)\Big] \\
&\to H_\lambda(x,y)
\end{aligned}
$$

as $n \to \infty$. The desired result follows. □

The following result is useful in the proof of Lemma 3.3.

**Lemma 3.2.** *With $a_n = 1/b_n$, for large $n$ we have*

$$
\begin{aligned}
& \int_{y}^{\infty} \Phi\left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \exp\left(\frac{b_n^2}{2}\left(1 - (1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1}\, dz \\
&= \int_{y}^{\infty} \Phi\left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right)\left(1 + \left((1-t)z - \frac{2-t}{2}z^2\right) b_n^{-2}\right) e^{-z} dz + O(b_n^{-4}). \qquad (3.8)
\end{aligned}
$$

**Proof.** First note that for large $n$ and $|x| \le \frac{b_n^2}{4(4+t)}$,

$$
\begin{aligned}
& \left| \exp\left(\frac{b_n^2}{2}\left(1 - (1+txa_n^2)^{2/t}\right)\right)(1+txa_n^2)^{1/t-1} - e^{-x}\left(1 + \frac{1}{b_n^2}\left((1-t)x - \frac{2-t}{2}x^2\right)\right)\right| \\
&\le b_n^{-4} s(x) \exp\left(-x + \frac{|x|}{4}\right), \qquad (3.9)
\end{aligned}
$$

where $a_n = 1/b_n$ and $s(x) \ge 0$ is a polynomial on $x$ independent of $n$, cf. Lemma 3.2 in [13].

It follows from (3.9) that

$$\begin{aligned}
&\int_y^{4\log b_n} \left| \Phi\left(\frac{\omega_{n,t}(x)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1} \right.\\
&\quad \left. -\Phi\left(\frac{\omega_{n,t}(x)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) e^{-z}\left(1+b_n^{-2}\left((1-t)z-\frac{2-t}{2}z^2\right)\right)\right| dz \\
&\le b_n^{-4}\int_y^{4\log b_n} s(z)\exp\left(-\frac{3z}{4}\right)dz \\
&= O(b_n^{-4}) \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{4\log b_n}^{\infty} e^{-z}\left(1+b_n^{-2}\left(|1-t|\,z+\frac{|2-t|}{2}z^2\right)\right)dz \\
&\le e^{-2\log b_n}\left(1+b_n^{-2}\left(4\,|1-t|\log b_n+8\,|2-t|\,(\log b_n)^2\right)\right)\int_{4\log b_n}^{\infty} e^{-\frac{z}{2}}dz \\
&= 2b_n^{-4}\left(1+b_n^{-2}\left(4\,|1-t|\log b_n+8\,|2-t|\,(\log b_n)^2\right)\right) \\
&= O(b_n^{-4}). \qquad (3.11)
\end{aligned}$$

So, the remainder is to show

$$A_n = \int_{4\log b_n}^{\infty} \exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1}dz = O(b_n^{-4}) \qquad (3.12)$$

for large $n$. We check (3.12) in turn for $0<t<1$ and $t\ge 1$.

For $0<t<1$, separate $A_n$ into the following two parts.

$$\begin{aligned}
A_{n1} &= \int_{4\log b_n}^{2(\frac{1}{t}-1)b_n^2} \exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1}dz \\
&< \int_{4\log b_n}^{2(\frac{1}{t}-1)b_n^2} e^{-z}(1+2(1-t))^{\frac{1}{t}-1}dz \\
&= O(b_n^{-4}) \qquad (3.13)
\end{aligned}$$

since $\exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{2/t}\right)\right) < e^{-z}$. For the second part,

$$A_{n2} = \int_{2(1/t-1)b_n^2}^{\infty} \exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1}dz$$

$$< \int_{2(1/t-1)b_n^2}^{\infty} e^{-z}(tza_n^2)^{1/t-1}\left(1+\frac{1}{tza_n^2}\right)^{1/t-1} dz$$
$$< (ta_n^2)^{1/t-1}\left(1+\frac{1}{2(1-t)}\right)^{1/t-1}\int_{2(1/t-1)b_n^2}^{\infty} e^{-z}z^{1/t-1}dz$$
$$= 2(3-2t)^{1/t-1}e^{-2(1/t-1)b_n^2}$$
$$= o(b_n^{-4}). \tag{3.14}$$

Hence, (3.13) and (3.14) shows that (3.12) holds as $0 < t < 1$.

Now switch to the case of $t \geq 1$. By Mills' inequality, we have

$$A_n = \int_{4\log b_n}^{\infty} \exp\left(\frac{b_n^2}{2}\right)\exp\left(-\frac{b_n^2(1+\frac{tz}{b_n^2})^{2/t}}{2}\right)\left(1+\frac{tz}{b_n^2}\right)^{1/t-1} dz$$
$$= b_n \exp\left(\frac{b_n^2}{2}\right)\int_{b_n\left(1+\frac{4t\log b_n}{b_n^2}\right)^{1/t}}^{\infty} \exp\left(-\frac{s^2}{2}\right) ds$$
$$= \sqrt{2\pi}b_n \exp\left(\frac{b_n^2}{2}\right)\left(1-\Phi\left(b_n\left(1+\frac{4t\log b_n}{b_n^2}\right)^{1/t}\right)\right)$$
$$< \frac{\exp\left(\frac{b_n^2}{2}\left(1-\left(1+\frac{4t\log b_n}{b_n^2}\right)^{2/t}\right)\right)}{\left(1+\frac{4t\log b_n}{b_n^2}\right)^{1/t}}$$
$$< \frac{\exp\left(-4\log b_n+\frac{8(t-2)(\log b_n)^2}{b_n^2}\right)}{\left(1+\frac{4t\log b_n}{b_n^2}\right)^{1/t}}$$
$$= O(b_n^{-4})$$

since $(1+s)^{\frac{2}{t}} \geq 1+\frac{2}{t}s+\frac{1}{t}\left(\frac{2}{t}-1\right)s^2$ for $s > 0$. The claimed result (3.12) follows as $t \geq 1$.

Combining (3.10)–(3.12), the proof of (3.8) is complete. □

In order to show the second order asymptotic expansions of extreme value distributions, let

$$\tilde{\Delta}(F_{\rho_n,t}^n, H_\lambda; c_n, d_n; x, y) = \mathbb{P}\left(M_{n1} \leq \omega_{n,t}(x), M_{n2} \leq \omega_{n,t}(y)\right) - H_\lambda(x,y).$$

**Lemma 3.3.** *Assume that the conditions of Theorem 2.2 hold. Then,*

$$\lim_{n\to\infty}(\log n)\tilde{\Delta}(F_{\rho_n,t}^n, H_\lambda; c_n, d_n; x, y) = \frac{1}{2}\tau(\alpha,\lambda,x,y,t)H_\lambda(x,y), \tag{3.15}$$

*where $\tau(\alpha,\lambda,x,y,t)$ is given by Theorem 2.2.*

**Proof.** By (3.6) and (3.8), we have

$$
\begin{aligned}
& n\,\mathbb{P}\left(X > \omega_{n,t}(x), Y > \omega_{n,t}(y)\right) \\
&= \bar{\Phi}_{n,t}(y) - \int_y^\infty \Phi\left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) e^{-z} \left(1 + \left((1-t)z - \frac{2-t}{2} z^2\right) b_n^{-2}\right) dz + O(b_n^{-4})
\end{aligned}
$$

for large $n$. It follows from (3.4) and (3.5) that

$$
\begin{aligned}
& b_n^2 \int_y^\infty \left(\lambda + \frac{x-z}{2\lambda} - \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \varphi\left(\lambda + \frac{x-z}{2\lambda}\right) e^{-z} dz \\
&\to \left(\alpha - \frac{1}{2}\lambda^3 - \frac{1}{2}\alpha\lambda^{-2}x - \frac{1}{4}\lambda x - \frac{1-t}{4\lambda}x^2\right) I_0 - \left(\frac{3}{4}\lambda - \frac{1}{2}\alpha\lambda^{-2}\right) I_1 + \frac{1-t}{4\lambda} I_2 \\
&= \kappa_1(\alpha, \lambda, x, y, t)
\end{aligned} \tag{3.16}
$$

as $n \to \infty$, where $I_k$ is given by (3.2) and

$$
\begin{aligned}
& \kappa_1(\alpha, \lambda, x, y, t) \\
&= 2\Big((2-t)\lambda^4 - (2-t)\lambda^2 x + (1-t)\lambda^2\Big) \bar{\Phi}\left(\lambda + \frac{y-x}{2\lambda}\right) e^{-x} \\
&\quad + \Big(2\alpha - (5-2t)\lambda^3 + (1-t)\lambda x + (1-t)\lambda y\Big) \varphi\left(\lambda + \frac{y-x}{2\lambda}\right) e^{-x}.
\end{aligned}
$$

Note that by Taylor's expansion with Lagrange remainder term,

$$
\begin{aligned}
& \Phi\left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \\
&= \Phi\left(\lambda + \frac{x-z}{2\lambda}\right) + \varphi\left(\lambda + \frac{x-z}{2\lambda}\right) \left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda}\right) \\
&\quad + \frac{1}{2} v_n \varphi(v_n) \left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda}\right)^2,
\end{aligned} \tag{3.17}
$$

where $v_n$ is between $\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}$ and $\lambda + \frac{x-z}{2\lambda}$. By arguments similar to (3.16), one can check that

$$
\int_y^\infty \left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda}\right)^2 v_n \varphi(v_n) e^{-z} dz = O(b_n^{-4}) \tag{3.18}
$$

holds for large $n$. Hence from (3.16), (3.17) and (3.18), it follows that

$$
\lim_{n\to\infty} b_n^2 \int_y^\infty \left(\Phi\left(\lambda + \frac{x-z}{2\lambda}\right) - \Phi\left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right)\right) e^{-z} dz = \kappa_1(\alpha, \lambda, x, y, t). \tag{3.19}
$$

Note that

$$
\bar{\Phi}_{n,t}(x) = e^{-x} - b_n^{-2}\mu(x) + O(b_n^{-4}), \tag{3.20}
$$

cf. Theorem 1 in [3]. Now combining (3.6), (3.19) and (3.20), we have

$$
\begin{aligned}
& b_n^2\Big[\mathbb{P}\left(M_{n1} \le \omega_{n,t}(x), M_{n2} \le \omega_{n,t}(y)\right) - H_\lambda(x,y)\Big] \\
&= b_n^2 H_\lambda(x,y)(1+o(1))\Bigg[-\bar{\Phi}_{n,t}(x) - \bar{\Phi}_{n,t}(y) + n\,\mathbb{P}(X > \omega_{n,t}(x), Y > \omega_{n,t}(y)) \\
&\quad + \Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y} + \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x}\Bigg] \\
&= b_n^2 H_\lambda(x,y)(1+o(1))\Bigg[-\bar{\Phi}_{n,t}(x) + e^{-x} + \int_y^\infty \Phi\left(\lambda + \frac{x-z}{2\lambda}\right)e^{-z}dz \\
&\quad - \int_y^\infty \Phi\left(\frac{\omega_{n,t}(x) - \rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\left(1 + \left((1-t)z - \frac{2-t}{2}z^2\right)b_n^{-2}\right)dz + O(b_n^{-4})\Bigg] \\
&\to H_\lambda(x,y)\Big[\mu(x) + \kappa_1(\alpha,\lambda,x,y,t) - \kappa_2(\alpha,\lambda,x,y,t)\Big] \\
&= H_\lambda(x,y)\tau(\alpha,\lambda,x,y,t)
\end{aligned}
$$

as $n \to \infty$, where

$$
\begin{aligned}
& \kappa_2(\alpha,\lambda,x,y,t) \\
&= \int_y^\infty \Phi\left(\lambda + \frac{x-z}{2\lambda}\right)e^{-z}\left((1-t)z - \frac{2-t}{2}z^2\right)dz \\
&= -\left(\frac{2-t}{2}y^2 + y + 1\right)\Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y} \\
&\quad + \left(2(2-t)\lambda^4 - 2(2-t)\lambda^2 x + 2(1-t)\lambda^2 + \frac{2-t}{2}x^2 + x + 1\right)\bar{\Phi}\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x} \\
&\quad - \Big(2(2-t)\lambda^3 - (2-t)\lambda(x+y) - 2\lambda\Big)\varphi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x}
\end{aligned}
$$

and $\tau(\alpha,\lambda,x,y,t)$ is given by Theorem 2.2. The proof is complete. □

**Lemma 3.4.** *Let the normalizing constants $c_n$ and $d_n$ be given by (1.7). For $\rho_n \in [-1,1)$,*

(i) *if $\rho_n \in [-1,0]$, we have*

$$
\lim_{n\to\infty}(\log n)\tilde{\Delta}(F^n_{\rho_n,t}, H_\infty; c_n, d_n; x, y) = \frac{1}{2}\Big(\mu(x) + \mu(y)\Big)H_\infty(x,y), \tag{3.21}
$$

(ii) *if $\rho_n \in (0,1)$ and $\lim_{n\to\infty}\frac{\log b_n}{b_n^2(1-\rho_n)} = 0$, then (3.21) also holds.*

**Proof.** (i). Note that complete independence ($\rho_n \equiv 0$) and complete negative dependence ($\rho_n \equiv -1$) imply $\lambda = \infty$. It follows from (3.20) that both

$$
\begin{aligned}
& b_n^2\left(-n(1 - F_{-1}(\omega_{n,t}(x), \omega_{n,t}(y))) + e^{-y} + e^{-x}\right) \\
&= b_n^2\left(-\bar{\Phi}_{n,t}(x) + e^{-x}\right) + b_n^2\left(-\bar{\Phi}_{n,t}(y) + e^{-y}\right) + nb_n^2\,\mathbb{P}(\omega_{n,t}(x) < X < -\omega_{n,t}(y)) \\
&\to \mu(x) + \mu(y)
\end{aligned} \tag{3.22}
$$

and

$$\begin{aligned}
&b_n^2\left(-n(1-F_0(\omega_{n,t}(x),\omega_{n,t}(y)))+e^{-y}+e^{-x}\right)\\
&=b_n^2\left(-\bar{\Phi}_{n,t}(x)+e^{-x}\right)+b_n^2\left(-\bar{\Phi}_{n,t}(y)+e^{-y}\right)+\frac{b_n^2}{n}\bar{\Phi}_{n,t}(x)\bar{\Phi}_{n,t}(y)\\
&\to\mu(x)+\mu(y)
\end{aligned}\tag{3.23}$$

hold as $n\to\infty$, showing that the claimed results (3.21) hold for $\rho_n\equiv -1$ and $\rho_n\equiv 0$ respectively. Thus, it follows from Slepian's Lemma that (3.21) also holds for $\rho_n\in[-1,0]$.

(ii). Obviously, $\lim_{n\to\infty}\frac{\log b_n}{b_n^2(1-\rho_n)}=0$ implies $\lim_{n\to\infty}\lambda_n=\infty$. Hence, for fixed $x,z\in\mathbb{R}$, one can check that

$$\lim_{n\to\infty}\frac{\omega_{n,t}(x)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}=\infty.\tag{3.24}$$

By (3.24) and Mills' inequality,

$$\begin{aligned}
&b_n^4\left(1-\Phi\left(\frac{\omega_{n,t}(x)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right)\right)\\
&<\frac{b_n^4\exp\left(-\frac{(\omega_{n,t}(x)-\rho_n\omega_{n,t}(z))^2}{2(1-\rho_n^2)}\right)}{\frac{\omega_{n,t}(x)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}}\\
&=\frac{\exp\left(-\frac{((c_nx+d_n)^{1/t}-\rho_n(c_nz+d_n)^{1/t})^2}{2(1-\rho_n^2)}+4\log b_n\right)}{\frac{(c_nx+d_n)^{1/t}-\rho_n(c_nz+d_n)^{1/t}}{\sqrt{1-\rho_n^2}}}\\
&=\left(b_n\sqrt{\frac{1-\rho_n}{1+\rho_n}}+\frac{x-z}{b_n\sqrt{1-\rho_n^2}}+\frac{z}{b_n}\sqrt{\frac{1-\rho_n}{1+\rho_n}}+\frac{(1-t)(x^2-\rho_nz^2)}{2b_n^3\sqrt{1-\rho_n^2}}+\sqrt{\frac{1-\rho_n}{1+\rho_n}}O(b_n^{-5})\right)^{-1}\\
&\quad\times\exp\left(-\frac{b_n^2(1-\rho_n)}{2(1+\rho_n)}-\frac{(x-\rho_nz)^2}{2b_n^2(1-\rho_n^2)}-\frac{(1-t)^2(x^2-\rho_nz^2)^2}{8b_n^6(1-\rho_n^2)}-\frac{x-\rho_nz}{1+\rho_n}-\frac{(1-t)(x^2-\rho_nz^2)}{2b_n^2(1+\rho_n)}\right.\\
&\quad\left.-\frac{(x-\rho_nz)(1-t)(x^2-\rho_nz^2)}{2b_n^4(1-\rho_n^2)}+\frac{1-\rho_n}{2(1+\rho_n)}O(b_n^{-5})+4\log b_n\right)\\
&<(1+o(1))e^{-\frac{x-z}{2}}\exp\left\{-\frac{b_n^2(1-\rho_n)}{2(1+\rho_n)}\left(1-\frac{8(1+\rho_n)\log b_n}{b_n^2(1-\rho_n)}+\frac{(1+\rho_n)\log b_n^2(1-\rho_n)}{b_n^2(1-\rho_n)}\right)\right\}\\
&\to 0
\end{aligned}\tag{3.25}$$

as $n\to\infty$. Note that

$$n^{-1}=\bar{\Phi}(b_n)=\frac{\varphi(b_n)}{b_n}(1-b_n^{-2}+O(b_n^{-4})).\tag{3.26}$$

Hence, by (3.10)–(3.12) and (3.26), we have

$$\begin{aligned}
&n\,\mathbb{P}(X>\omega_{n,t}(x),Y>\omega_{n,t}(y))\\
&=b_n^{-4}(1-b_n^{-2}+O(b_n^{-4}))^{-1}\int_y^\infty b_n^4\bar{\Phi}(\frac{\omega_{n,t}(x)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}})\exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1}dz\\
&=O(b_n^{-4}).
\end{aligned}\tag{3.27}$$

It follows from (3.20) and (3.27) that

$$\begin{aligned}
& b_n^2\Big[F_{\rho_n}^n(\omega_{n,t}(x),\omega_{n,t}(y)) - H_\infty(x,y)\Big] \\
&= b_n^2 H_\infty(x,y)(1+o(1))\Big[-n(1-F_{\rho_n}(\omega_{n,t}(x),\omega_{n,t}(y))) + e^{-x} + e^{-y}\Big] \\
&= b_n^2 H_\infty(x,y)(1+o(1))\Big[-(e^{-x}+e^{-y}-b_n^{-2}(\mu(x)+\mu(y)+O(b_n^{-2}))) + e^{-x}+e^{-y}\Big] \\
&\to H_\infty(x,y)\Big[\mu(x)+\mu(y)\Big]
\end{aligned}$$

as $n\to\infty$. The proof is complete. □

**Lemma 3.5.** *Let the normalizing constants $c_n$ and $d_n$ be given by (1.7). For $\rho_n \in (0,1]$,*

(i) *if $\rho_n \equiv 1$, we have*

$$\lim_{n\to\infty}(\log n)\tilde{\Delta}(F_{\rho_n,t}^n, H_0; c_n, d_n; x, y) = \frac{1}{2}\mu\Big(\min(x,y)\Big)H_0(x,y), \tag{3.28}$$

(ii) *if $\rho_n \in (0,1)$ and $\lim_{n\to\infty} b_n^6(1-\rho_n) = c_1 \in [0,\infty)$, then (3.28) also holds.*

**Proof.** (i). Note that the complete positive dependent case $\rho_n \equiv 1$ implies $\lambda = 0$. It follows from (3.20) that

$$b_n^2\Big[-n(1-F_1(\omega_{n,t}(x),\omega_{n,t}(y))) + e^{-\min(x,y)}\Big] = b_n^2\Big[-\bar{\Phi}_{n,t}(\min(x,y)) + e^{-\min(x,y)}\Big] \to \mu(\min(x,y)) \tag{3.29}$$

as $n\to\infty$. It follows from (3.29) that (3.28) holds as $\rho_n \equiv 1$.

(ii). Without loss of generality, assume that $y < x \in \mathbb{R}$. For $\max(x,y) = x < z < 4\log b_n$ we have

$$\begin{aligned}
& \Phi\left(\frac{\omega_{n,t}(y)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \\
&< -\frac{\varphi\left(\frac{\omega_{n,t}(y)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right)}{\frac{\omega_{n,t}(y)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}} \\
&\le \frac{\exp\left(-\frac{1}{2}\left(\lambda_n + \frac{y-z}{2\lambda_n}\right)^2(1+o(1))\right)}{\left(-\lambda_n + \frac{z-y}{2\lambda_n}\left(1+\frac{(1-t)(y+z)}{2b_n^2}\right) - \frac{\lambda_n z}{b_n^2} - \frac{(1-t)\lambda_n z^2}{2b_n^4} + \lambda_n O(b_n^{-6})\right)(1-\frac{\lambda_n^2}{b_n^2})^{-\frac{1}{2}}} \\
&= \frac{\exp\left(-\frac{1}{2}\left(\lambda_n + \frac{y-z}{2\lambda_n}\right)^2(1+o(1))\right)}{\frac{z-y}{2\lambda_n}(1+o(1))}
\end{aligned} \tag{3.30}$$

for large $n$ due to $\Phi(-x) = \bar{\Phi}(x)$ and Mills' inequality since $\frac{\omega_{n,t}(y)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} < 0$ for large $n$ when $\lim_{n\to\infty} b_n^6(1-\rho_n) = c_1 \in [0,\infty)$. Therefore,

$$\int_x^{4\log b_n} \Phi\left(\frac{\omega_{n,t}(y)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right)\exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1}dz$$

$$
\begin{aligned}
&< \frac{2\lambda_n}{x-y}(1+o(1)) \int_x^{4\log b_n} \exp\left(-\frac{\lambda_n^2}{2} - \frac{y-z}{2} - \frac{(y-z)^2}{8\lambda_n^2} - z + o(b_n^{-1}) + \left(\frac{1}{t}-1\right)\log(1+tza_n^2)\right) dz \\
&= 2\lambda_n(1+o(1)) \frac{\exp\left(-\frac{\lambda_n^2}{2} - \frac{y}{2} - \frac{(y-x)^2}{8\lambda_n^2}\right)}{x-y} \int_x^{4\log b_n} \exp\left(-\frac{z}{2}\right) dz \\
&< 4\lambda_n b_n^{-2}(1+o(1)) \frac{\exp\left(-\frac{\lambda_n^2}{2} - \frac{y}{2} - \frac{(y-x)^2}{8\lambda_n^2}\right)}{y-x} \\
&= O(b_n^{-4})
\end{aligned} \tag{3.31}
$$

for large $n$ by $\lim_{n\to\infty} b_n^6(1-\rho_n) = c_1$. It follows from (3.12) that

$$
\begin{aligned}
&\int_{4\log b_n}^{\infty} \Phi\left(\frac{(\omega_{n,t}(y) - \rho_n\omega_{n,t}(z))^2}{\sqrt{1-\rho_n^2}}\right) \exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1} dz \\
&< \int_{4\log b_n}^{\infty} \exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{2/t}\right)\right)(1+tza_n^2)^{1/t-1} dz \\
&= O(b_n^{-4}).
\end{aligned} \tag{3.32}
$$

Combining (3.20), (3.31) and (3.32), for $y < x$ we have

$$
\begin{aligned}
&1 - F_{\rho_n}\left(\omega_{n,t}(\min(x,y)), \omega_{n,t}(\max(x,y))\right) \\
&= \bar{\Phi}_{n,t}(y) + \bar{\Phi}_{n,t}(x) - \mathbb{P}\left(X > \omega_{n,t}(y), Y > \omega_{n,t}(x)\right) \\
&= \bar{\Phi}_{n,t}(y) + \bar{\Phi}_{n,t}(x) \\
&\quad - \int_x^{\infty} \bar{\Phi}\left(\frac{\omega_{n,t}(y) - \rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{\frac{2}{t}}\right)\right)(1+tza_n^2)^{\frac{1}{t}-1} dz \\
&= \bar{\Phi}_{n,t}(y) + n^{-1}(1 - b_n^{-2} + O(b_n^{-4}))^{-1} \\
&\quad \times \int_x^{\infty} \Phi\left(\frac{\omega_{n,t}(y) - \rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \exp\left(\frac{b_n^2}{2}\left(1-(1+tza_n^2)^{\frac{2}{t}}\right)\right)(1+tza_n^2)^{\frac{1}{t}-1} dz \\
&= n^{-1}(e^{-y} - b_n^{-2}\mu(y) + O(b_n^{-4}))
\end{aligned}
$$

for large $n$, which implies the desired result. The proof is complete. □

**Lemma 3.6.** *With power index $t = 2$, and the normalizing constants $c_n^*$ and $d_n^*$ given by (1.8). For $\rho_n \in [-1, 1)$,*

(i) *if $\rho_n \in [-1, 0]$, we have*

$$
\lim_{n\to\infty} (\log n)^2 \tilde{\Delta}(F_{\rho_n,2}^n, H_\infty; c_n^*, d_n^*; x, y) = \frac{1}{4}\left(\nu(x) + \nu(y)\right) H_\infty(x,y), \tag{3.33}
$$

(ii) *if $\rho_n \in (0, 1)$ and $\lim_{n\to\infty} \frac{\log b_n}{b_n^2(1-\rho_n)} = 0$, then (3.33) also holds.*

**Proof.** (i). Note that

$$\bar{\Phi}_{n,2}(x) = e^{-x} - b_n^{-4}\nu(x) + O(b_n^{-6}) \tag{3.34}$$

derived by Theorem 1 in [3]. It follows from (3.34) that

$$b_n^4\left(-n(1 - F_{-1}(\omega_{n,2}^*(x), \omega_{n,2}^*(y))) + e^{-y} + e^{-x}\right) \to \nu(x) + \nu(y)$$

as $n \to \infty$ if $\rho_n \equiv -1$, and

$$b_n^4\left(-n(1 - F_0(\omega_{n,2}^*(x), \omega_{n,2}^*(y))) + e^{-y} + e^{-x}\right) \to \nu(x) + \nu(y)$$

also holds as $n \to \infty$ if $\rho_n \equiv 0$. Therefore, (3.33) holds for $\rho_n \equiv -1$ and $\rho_n \equiv 0$ respectively. By Slepian's Lemma, (3.33) also holds for $\rho_n \in [-1, 0]$.

(ii). Obviously, $\lim_{n\to\infty} \frac{\log b_n}{b_n^2(1-\rho_n)} = 0$ implies $\lim_{n\to\infty} \lambda_n = \infty$. Hence, for fixed $x$ and $z$,

$$\frac{\omega_{n,2}^*(x) - \rho_n\omega_{n,2}^*(z)}{\sqrt{1-\rho_n^2}} \to \infty$$

as $n \to \infty$. By arguments similar to (3.25), we have

$$b_n^6\left(1 - \Phi\left(\frac{\omega_{n,2}^*(x) - \rho_n\omega_{n,2}^*(z)}{\sqrt{1-\rho_n^2}}\right)\right) \to 0 \tag{3.35}$$

as $n \to \infty$. It follows from (3.26) and (3.35) that

$$\begin{aligned}
&\mathbb{P}(X > \omega_{n,2}^*(x), Y > \omega_{n,2}^*(y)) \\
&= n^{-1}b_n^{-6}(1 + b_n^{-2} + O(b_n^{-4}))\int_y^\infty b_n^6\left(1 - \Phi(\frac{\omega_{n,2}(x) - \rho_n\omega_{n,2}(y)}{\sqrt{1-\rho_n^2}})\right) \\
&\quad \times \exp\left(-z + (1+z)a_n^2\right)\left(1 - a_n^2\right)\left(1 + 2\left(z - (1+z)a_n^2\right)a_n^2\right)^{-\frac{1}{2}} dz \\
&= O(n^{-1}b_n^{-6})
\end{aligned}$$

for large $n$. Hence,

$$b_n^4\Big[F_{\rho_n}^n(\omega_{n,2}^*(x), \omega_{n,2}^*(y)) - H_\infty(x,y)\Big] \to H_\infty(x,y)\Big[\nu(x) + \nu(y)\Big]$$

as $n \to \infty$. The proof is complete. □

**Lemma 3.7.** *With power index $t = 2$, and the normalizing constants $c_n^*$ and $d_n^*$ given by (1.8). For $\rho_n \in (0, 1]$,*

(i) *if $\rho_n \equiv 1$, we have*

$$\lim_{n\to\infty}(\log n)^2\tilde{\Delta}(F_{\rho_n,2}^n, H_0; c_n^*, d_n^*; x, y) = \frac{1}{4}\nu\Big(\min(x,y)\Big)H_0(x,y), \tag{3.36}$$

(ii) *if $\rho_n \in (0, 1)$ and $\lim_{n\to\infty} b_n^{14}(1-\rho_n) = c_2 \in [0, \infty)$, then (3.36) also holds.*

**Proof.** (i). For the complete positive dependent case $\rho_n \equiv 1$, without loss of generality, assume that $y < x$. It follows from (3.34) that

$$b_n^4\Big[-n(1-F_1(\omega_{n,2}^*(x), \omega_{n,2}^*(y))) + e^{-y}\Big] \to \nu(y)$$

as $n \to \infty$. The desired result (3.36) follows as $\rho_n \equiv 1$.

(ii). By arguments similar to that of (3.31) and (3.32), for fixed $y < x \in \mathbb{R}$ we have

$$\int_x^\infty \Phi\left(\frac{\omega_{n,2}^*(y) - \rho_n \omega_{n,2}^*(z)}{\sqrt{1-\rho_n^2}}\right) \exp(-z + (1+z)a_n^2) \frac{1-a_n^2}{(1+2(z-(1+z)a_n^2)a_n^2)^{\frac{1}{2}}} dz = O(b_n^{-6}) \tag{3.37}$$

for large $n$ by $\lim_{n\to\infty} b_n^{14}(1-\rho_n) = c_2$. Combining (3.34) with (3.37), we have

$$\begin{aligned}
&1 - F_{\rho_n}(\omega_{n,2}^*(x), \omega_{n,2}^*(y)) \\
&= \bar{\Phi}_{n,2}(y) + n^{-1}(1 - b_n^{-2} + O(b_n^{-4}))^{-1} \\
&\quad \times \int_x^\infty \Phi\left(\frac{\omega_{n,2}^*(y) - \rho_n \omega_{n,2}^*(z)}{\sqrt{1-\rho_n^2}}\right) \exp(-z + (1+z)a_n^2) \frac{1-a_n^2}{(1+2(z-(1+z)a_n^2)a_n^2)^{\frac{1}{2}}} dz \\
&= n^{-1}\left(e^{-y} - b_n^{-4}\nu(y) + O(b_n^{-6})\right)
\end{aligned}$$

for large $n$, which implies the desired result. The proof is complete. □

## 4. Proofs

**Proof of Theorem 2.1.** Obviously,

$$\begin{aligned}
&\mathbb{P}\left(|M_{n1}|^t \le c_n x + d_n, |M_{n2}|^t \le c_n y + d_n\right) \\
&= F_{\rho_n,t}^n(\omega_{n,t}(x), \omega_{n,t}(y)) - F_{\rho_n,t}^n(\omega_{n,t}(x), -\omega_{n,t}(y)) - F_{\rho_n,t}^n(-\omega_{n,t}(x), \omega_{n,t}(y)) + F_{\rho_n,t}^n(-\omega_{n,t}(x), -\omega_{n,t}(y)).
\end{aligned}$$

Note that

$$\begin{aligned}
&F_{\rho_n,t}^n(\omega_{n,t}(x), -\omega_{n,t}(y)) + F_{\rho_n,t}^n(-\omega_{n,t}(x), \omega_{n,t}(y)) - F_{\rho_n,t}^n(-\omega_{n,t}(x), -\omega_{n,t}(y)) \\
&\le \mathbb{P}(M_{n2} \le -\omega_{n,t}(y)) + \mathbb{P}(M_{n1} \le -\omega_{n,t}(x)) - \min\{\Phi^n(-\omega_{n,t}(x)), \Phi^n(-\omega_{n,t}(y))\} \\
&= \Phi^n(-\omega_{n,t}(x)) + \Phi^n(-\omega_{n,t}(y)) - \min\{\Phi^n(-\omega_{n,t}(x)), \Phi^n(-\omega_{n,t}(y))\} \\
&= o(b_n^{-4})
\end{aligned} \tag{4.1}$$

since

$$\bar{\Phi}^{n-1}(-\omega_{n,t}(x)) = \left(n^{-1}e^{-x}(1 + O(b_n^{-2}))\right)^{n-1} = o(b_n^{-4}),$$

cf. Lemma 3.1 in [17]. Combining (4.1) with Lemma 3.1, we can get the desired result (2.1). □

**Proof of Theorem 2.2.** It follows from (4.1) and Lemma 3.3 that

$$\Delta(F_{\rho_n,t}^n, H_\lambda; c_n, d_n; x, y) = \tilde{\Delta}(F_{\rho_n,t}^n, H_\lambda; c_n, d_n; x, y) + o(b_n^{-4}),$$

so the result (2.3) is obtained. □

**Proof of Theorem 2.3 and Theorem 2.4.** It follows from (4.1), Lemma 3.4 and Lemma 3.5, respectively. □

**Proof of Theorem 2.5 and Theorem 2.6.** By arguments similar to the proof of Theorem 2.1, we have

$$\Delta(F^n_{\rho_n,2}, H_\lambda; c^*_n, d^*_n; x, y) = \tilde{\Delta}(F^n_{\rho_n,2}, H_\lambda; c^*_n, d^*_n; x, y) + o(b_n^{-6}),$$

so the desired results follow from Lemma 3.6 and Lemma 3.7, respectively. □

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